

Pure Fulde-Ferrell-Larkin-Ovchinnikov state in optical lattices

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We study the phase diagram of one-dimensional, two-component attractive fermions on optical lattices with an off-diagonal confinement. We identify in this system a pure Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state with spatially modulated superfluidity and polarization in a large parameter window, which provides an easier playground to detect the FFLO state experimentally. The clear signature of FFLO state is analyzed in the ground state properties, the pair correlation function, and the magnetic structure factor, using the density-matrix renormalization group method.

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Introduction. —Experimental developments in cold atomic gases provide a promising testbed to realize the exotic quantum phases predicted in condensed matter systems (for details see Ref. [1] for a recent review). Among these phases, there is an important state first predicted by Fulde, Ferrell, Larkin, and Ovchinnikov (FFLO) in 1964 for superconductors in high magnetic fields, characterized by Cooper pairs with nonzero center-of-mass momentum and a spatially inhomogeneous order parameter.²

Though the FFLO state has not been explicitly confirmed experimentally, its realization in ultracold atomic systems is becoming promising due to the great adjustability for almost all of the system parameters, for example, the spin imbalance realized by the magnetic fields in solid state physics can be easily simulated using the mismatched different particle species by radio-frequency fields.³ While the FFLO state is difficult to detect in three-dimensional ultracold atomic systems due to the very small parameter window in free space,⁴ the possibility of direct realization in the one-dimensional (1D) case increases due to the major part of the phase diagram being of the character of the FFLO correlation, based on analytical,^{5,6} numerical,⁷ and experimental⁸ research.

Experimentally⁸ the density profiles of a spin imbalanced mixture of ultracold ⁶Li atoms trapped in an array of 1D tubes are measured with the standard absorption imaging. The density profiles detected meet the criteria of the FFLO state predicted analytically.⁶ In a 1D system of harmonic confinement, a partially polarized core is surrounded by wings of either a fully paired or a fully polarized Fermi gas depending on the controlled imbalanced population in its two spin states. The partially polarized core in the density profile is proved of FFLO characteristics by calculating the pair momentum.⁶ But in order to demonstrate directly that this pair state is of FFLO type, one needs to precisely measure the physical quantities peculiar to FFLO states, for example, the condensed fraction using the time-of-flight (TOF) measurement and the quantities other than density profile revealing the finite momentum of the pairing order parameter, like the correlations in the shot noise of fermion absorption images in TOF.^{3,9,10} In a 1D optical lattice under the presence of a harmonic confining potential, Korolyuk et al. proposed that the change of the double occupancy due to the lattice depth modulation provides a clear evidence.¹¹ The collective modes produced by the response of the ground state to time-dependent potentials give

another possible signature of the FFLO phase.^{12,13} Very recently, Kajala et al. proved that the TOF expansion manifests a direct signature of the FFLO state in the 1D system in which the expansion velocity of the unpaired particles matches the expected FFLO momentum.¹⁴

In cold atom experiments, the necessary confining potential, which is usually a single-particle potential called diagonal confinement, leads to the emergence of coexisting states¹⁵ and inevitably makes the detected signals complicated. In the repulsive case the diagonal confinement leads to the absence of true incompressible Mott insulating phases and a real superfluid-to-Mott transition.¹⁶ Though Scarola et al. pointed out that edge effects in a trapped Fermi-Hubbard system can be filtered out with a direct probe independent of inhomogeneity¹⁷ by relating the core compressibility to changes in the double occupancy, people hope to realize directly a pure Mott insulating or other exotic phases in the trapped system by making use of state-of-the-art cold atomic techniques. In order to achieve such a goal, an off-diagonal confinement method was pointed out by Rousseau et al.,^{16,18} where site-dependent hopping integrals are designed with amplitude decreasing while moving away from the center of the lattice and vanishing towards the boundaries.

In this paper, we show that for the system in the off-diagonal confinement of population imbalance, a pure FFLO state is realized at a large range of system parameters, and thus, in this case, the common multi-phase structure in the diagonal confinement system is avoided, which facilitates the detection of such an elusive state in the cold atomic system. The clear signature of FFLO state is analyzed through calculating the pair correlation function and the magnetic structure factor, using density-matrix renormalization group (DMRG) method.

The model and phase diagram. —We consider fermions confined to a 1D optical lattice with L sites and lattice constant $a = 1$. The Hamiltonian is given by

$$H = - \sum_{\langle i,j \rangle, \sigma} t_{ij} (c_{i,\sigma}^\dagger c_{j,\sigma} + h.c.) + U \sum_{i=0}^{L-1} n_{i\uparrow} n_{i\downarrow}, \quad (1)$$

where $c_{i,\sigma}^{(+)}$ is a fermionic annihilation (creation) operator acting on site i with pseudospin σ representing the hyperfine states and the sum over $\langle i, j \rangle$ is restricted to the nearest neigh-

bours. The spatially varying tunneling is of inverted parabola designed as

$$t_{ij} = t(i+j+1)(2L-i-j-1)/L^2, \quad (2)$$

which vanish at the edges avoiding the escape of the particles from the edges, namely, $t_{-1,0} = t_{L-1,L} = 0$. The properties of the FFLO state in a 1D optical lattice of mass imbalance have been studied intensively,^{19–21} where t_{ij} in Eq. (1) is replaced by t_σ . They found that the typical FFLO modulation is preserved in the gapped phase but not the quasi-long-range order¹⁹ and if the system is partially polarized, the ground-state FFLO state is more stable in the presence of mass imbalance.²¹

We study the case of attractive on-site interactions ($U < 0$) with population imbalance. In the following, we take $u = U/t$ with t as the units of energy. The number of fermions of each species is $N_\sigma = \sum_i \langle c_{i,\sigma}^\dagger c_{i,\sigma} \rangle$ with $N = N_\uparrow + N_\downarrow$ and the spin $\sigma = \uparrow, \downarrow$. The filling factor, the magnetization, and the polarization are $n = N/L$, $S_z = (N_\uparrow - N_\downarrow)/2$, and $p = (N_\uparrow - N_\downarrow)/N$, respectively. We work in the canonical ensemble with fixed total number of particles. In the following we choose a typical experimentally accessible optical lattice of $L = 70$ sites and assume $N_\uparrow > N_\downarrow$ without loss of generality.

Experimentally, this off-diagonal confinement system can be realized by a holographic method with a mask under the present-day ultracold atomic techniques.¹⁶ Technically, the off-diagonal confinement system is somehow equivalent to the smooth boundary conditions introduced by Vekić and White aiming at reducing finite-size effects and extrapolating to the thermodynamic limit on relatively small systems,²² and is similar to the sine-squared or sinusoidal deformation applied to 1D quantum Hamiltonians.²³ In our study we discuss a partially polarized system of inverted parabola tunneling.²⁴

The ground-state phase diagram is shown in Fig. 1. The typical density distributions in different ranges are displayed as insets. We find that a partially polarized phase ($0 < n_{i\uparrow}, n_{i\downarrow} < 1$) exists in the core of the system at finite polarization. Among all the different phases, we identify an important pure FFLO superfluid phase which is partially polarized in the whole space of the system in part I, for $S_z \in [8, 18]$ and $n \in [0.93, 1.1]$. We would like to point out that in the diagonal confinement system the pure FFLO state happens only at a certain critical polarization. Other phases identified are, a fully polarized phase ($0 < n_{i\uparrow} < 1, n_{i\downarrow} = 0$, FP), an unpolarized BCS phase ($n_{i\uparrow} = n_{i\downarrow} \neq 0$, BCS), a pure Fock state of localized pairs, namely a band-insulating state, at high densities ($n_{i\uparrow} = n_{i\downarrow} = 1$, FS), and fully polarized ferromagnetic insulating phase for one species ($n_{i\uparrow} = 1$) and partially filled ($n_{i\downarrow} \neq 0$, FIa) or empty ($n_{i\downarrow} = 0$, FIb) for another. All the regions in Fig. 1 except part I are composed of coexisting phases, either of two-phase structure: FFLO-FP (IIa, IIc), FFLO-FIa (IIId), and FFLO-BCS (IIIa, IIId), FFLO-FIb (IV), or of three-phase structure: FFLO-FI-FS (IIb) and FFLO-BCS-FS (IIId). The relation between the parts of IIa and IIb, IIc and IIId, and IIIa and IIId, can be understood by the particle-hole symmetry, $c_{i\sigma} \rightarrow (-1)^i c_{i\sigma}^\dagger$, mapping the

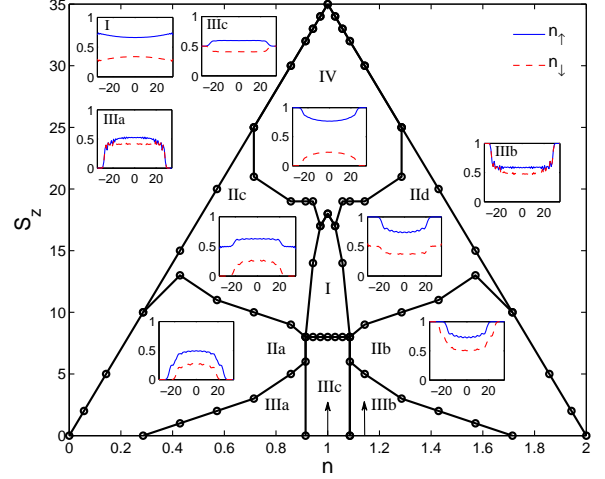


FIG. 1: (Color online) The ground-state phase diagram of the off-diagonal confinement system in the S_z - n plane for $u = -4$. The pure FFLO superfluid phase (I) locates in the parameter range of $8 < S_z < 18$ and $0.93 < n < 1.1$. Other composite phases we labeled are FFLO-FP (IIa, IIc), FFLO-FI-FS (IIb), FFLO-FIa (IIId), FFLO-BCS (IIIa, IIIc), FFLO-BCS-FS (IIIb), and FFLO-FIb (IV), respectively (details see text). The density distributions for spin-up and spin-down particles in the different regions are shown versus i for typical situations as insets. The arrows indicate the direction of the different phases while increasing the magnetization S_z for fixed fermions of $N = 70$ and 80 .

Hamiltonian (1) onto itself. As a result, if the particle number in part IIa and part IIb satisfies $N_\uparrow^a + N_\downarrow^b = N_\downarrow^a + N_\uparrow^b$ and $N^a + N^b = 2L$, the densities in the corresponding parts are related by $n_{i\sigma}^a = 1 - n_{i\bar{\sigma}}^b$. Here $\bar{\sigma} = -\sigma$.

Pairing correlations.—Let us now come to the discussion of the pairing properties. We first compute the pair correlation function, which is defined as,

$$G(i, j) = \langle \Delta_i^\dagger \Delta_j \rangle \quad (3)$$

with $\Delta_i = c_{i\uparrow} c_{i\downarrow}$. Its Fourier transform is given by

$$G_{\text{pair}}(k) = \frac{1}{2L} \sum_{i,j} G(i, j) e^{ik(i-j)}. \quad (4)$$

Feiguin and Heidrich-Meisner confirmed that for the attractive Hubbard model in a harmonic confinement the momentum pair distribution has peaks at the mismatch of the Fermi surfaces.²⁵ This quantity is displayed in Fig. 2 for varying magnetization from $S_z = 0$ to 30 .

For the unpolarized case, the ground state is a BCS state characterized by a sharp peak centered at momentum $k = 0$ in the pair momentum distribution $G_{\text{pair}}(k)$. For the population imbalanced Fermi system of two spin species, two different Fermi surfaces exist and the ground state is a 1D FFLO state with a finite center-of-mass momentum ($k \neq 0$), which can be understood as an order parameter of the system. In the homogeneous system, the momentum of the FFLO

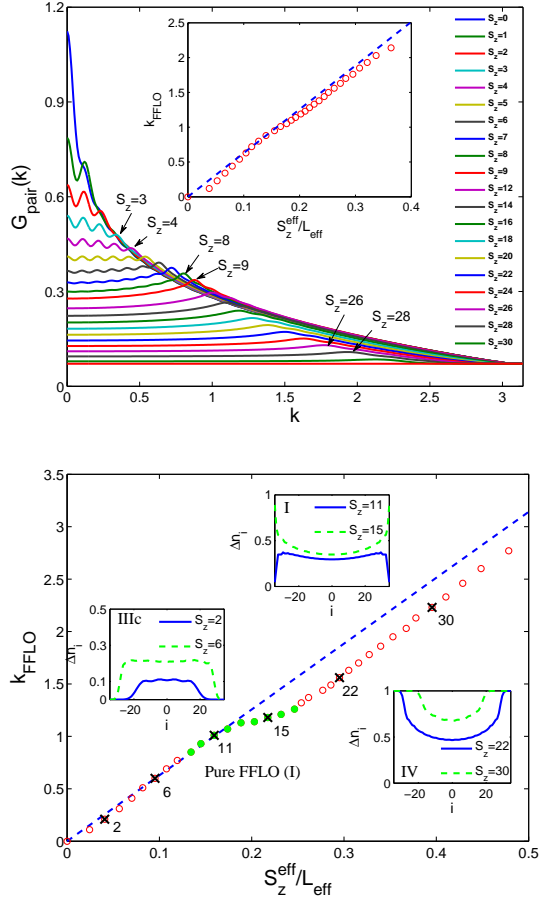


FIG. 2: (Color online) Upper panel: the pair momentum distributions for the magnetization varying from $S_z = 0$ to 30 (from top to bottom) in a system of $N = 80$ fermions. The inset shows that the momentum of the FFLO state k_{FFLO} , at which the G_{pair} is peaked, as a function of the effective magnetization at the core $S_z^{\text{eff}}/L_{\text{eff}}$. With the increase of S_z , the phases of the system change from IIIb \rightarrow IIb \rightarrow IIc \rightarrow IV for $N = 80$ and from IIIc \rightarrow I \rightarrow IV for $N = 70$ (in the lower panel), which is indicated in Fig. 1 by arrows. The dashed line represents $k_{\text{FFLO}} = \Delta \tilde{k}_F$. Lower panel: the position of the FFLO peak as a function of $S_z^{\text{eff}}/L_{\text{eff}}$ in a system of $N = 70$ fermions. k_{FFLO} in the pure FFLO region is denoted by the full circle. In the inset, the profiles of the populations difference Δn_i are shown for two different magnetizations S_z in the corresponding phase region. The crosses with S_z values are pertinent to the insets. See the text for a discussion of the relations between the linearity of k_{FFLO} and Δn_i in different phases. The interaction strength used here is $u = -4$.

state k_{FFLO} , at which the G_{pair} is peaked, is, $k_{\text{FFLO}} = k_{F\uparrow} - k_{F\downarrow} = \pi(n_{\uparrow} - n_{\downarrow}) = 2\pi S_z/L = \pi n p \neq 0$ with $k_{F\sigma} = \pi n_{\sigma} = \pi N_{\sigma}/L$. In the inhomogeneous system, we define $\Delta \tilde{k}_F = 2\pi S_z^{\text{eff}}/L_{\text{eff}}$, where S_z^{eff} is obtained by integrating $\langle S_z^i \rangle$ over the effective partially polarized FFLO region L_{eff} with S_z^i defined as $S_z^i = (n_{i\uparrow} - n_{i\downarrow})/2$.²⁵ We notice that, in Fig. 2, the value of k_{FFLO} increases with S_z as predicted by the FFLO picture. But in the inhomogeneous system

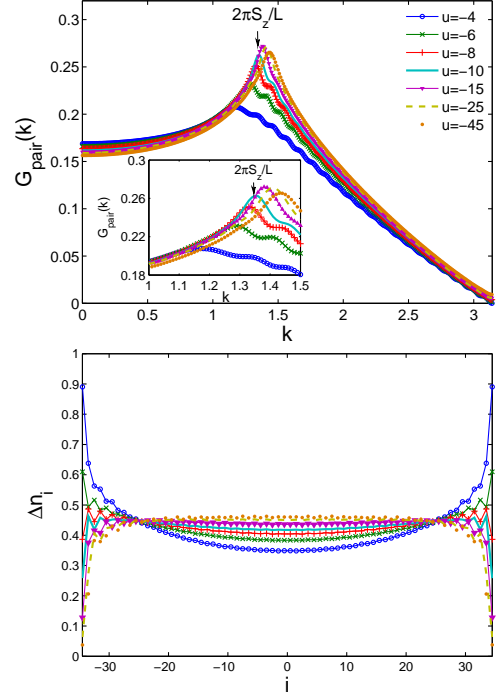


FIG. 3: (Color online) Upper panel: the pair momentum distribution for different interaction strength. The parameters used here are $S_z = 15$ and $N = 70$. The one in the lower-left magnifies the region around the position of the peaks. Lower panel: the corresponding density difference $\Delta n_i = n_{i\uparrow} - n_{i\downarrow}$ in the real space. The notations are the same as those in upper panel.

under the off-diagonal confinement, where the hopping amplitude t_{ij} is spatially varying, the relation between k_{FFLO} and $\Delta \tilde{k}_F$ is not as simple as that under the diagonal confinement. For $N = 80$, in the cases where the density differences in the bulk become inhomogeneous, k_{FFLO} deviates slightly from $\Delta \tilde{k}_F$. For the system of $N = 70$ (lower panel), the phases of the system undergoes IIIc \rightarrow I \rightarrow IV while increasing S_z . Starting from $S_z = 13$, k_{FFLO} greatly deviates from $\Delta \tilde{k}_F$ at which $\Delta n_i = n_{i\uparrow} - n_{i\downarrow}$ in the bulk becomes largely inhomogeneous. The FFLO peak around k_{FFLO} comes from the FFLO state in the bulk of the density, while the central peaks (of oscillating character) come from the paired BCS wings.

For the pure FFLO states, we show in Fig. 3 the effects of the interactions on the pair momentum distributions. We find the optimal interaction strength ($u \approx -15$) where the peak of G_{pair} reaches maximum. The position of the peak matches exactly the linearity relation of $k_{\text{FFLO}} = 2\pi S_z/L \approx 1.346$ at $u \approx -10$, where Δn_i is almost homogeneous, in the whole range of the system except at the edges.

Magnetic structure factor. — We further reveal fingerprints of the FFLO state by analyzing the magnetic structure factor (MSF), which is the Fourier transform of spin-spin correlations,

$$S_m(k) = \frac{1}{2L} \sum_{i,j} (\langle S_z^i S_z^j \rangle - \langle S_z^i \rangle \langle S_z^j \rangle) e^{ik(i-j)}. \quad (5)$$

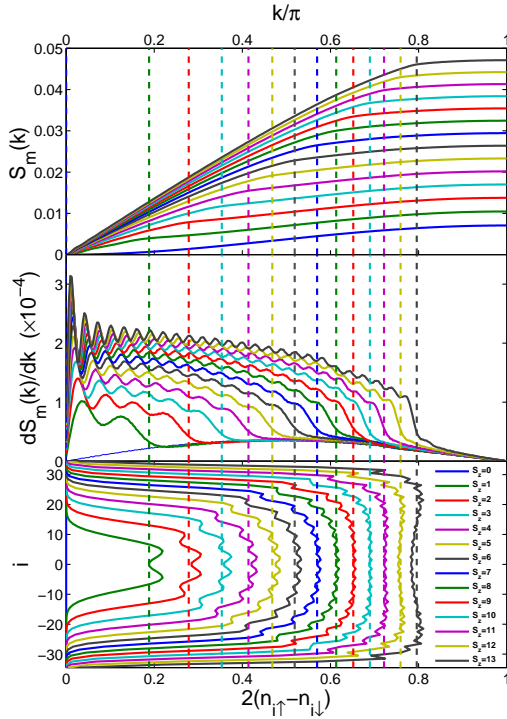


FIG. 4: (Color online) Upper panel: magnetic structure factor as a function of k/π with fixed number of fermions $N=70$ and interaction strength $u = -10$ for different magnetization S_z (from top $S_z = 0$ to bottom $S_z = 30$). Middle panel: the corresponding momentum derivative of the magnetic structure factor. Lower panel: corresponding density imbalance distributions, $2(n_{i\uparrow} - n_{i\downarrow})$. The dashed lines mark the matching positions of the kink in the structure factor, the corresponding jump in the derivative of the magnetic structure factor, and the central density imbalance.

Spin-spin correlations can be detected in a non-destructive way via spatially resolved quantum polarization spectroscopy. Roscilde et al. showed analytically and numerically that $2k_{\text{FFLO}}$ -modulation of spin-spin correlations in the MSF is a direct fingerprint of pairing in an imbalance system and serves as a good quantity in detecting the FFLO state.²⁶

We display the MSF in Fig. 4 for $N = 70$ and $u = -10$

with different polarization values. Comparing to the unpolarized case, there is an obvious kink in the MSF with finite S_z . To have a clear signature of this kink, we show the momentum derivative of the MSF which exhibits a marked jump, where the density imbalance distributions in the bulk, Δn_i , reveal a constant structure. The match of the location of the kink with the bulk region of the density deteriorates when the density difference becomes inhomogeneous, like the cases in the lower panel of Fig. 2 when $S_z > 13$. We notice that the density difference in the lower panel is obviously oscillatory for small polarizations. The wavelength of the oscillations is given by $\lambda = 2\pi/|k_{F\uparrow} - k_{F\downarrow}|$, which gives a confirmation that the FFLO state is inhomogeneous and exhibits real-space modulations.²⁷

Conclusions. — We have considered a 1D optical lattice of off-diagonal confinement, modeled by a single-band Hubbard model with spatially varying hopping amplitudes. We showed that with population imbalance the dominant pairing mechanism in the ground state is FFLO characterized by a finite center-of-mass momentum. We predicted based on the S_z - n phase diagram that a pure FFLO state exists at a large parameter range, greatly different from the diagonal confinement system where only a critical polarization can achieve that. We further revealed that the partially polarized densities are the results of the finite center-of-mass momentum pairing through calculating the pair correlation function and the spin-spin correlation. The flexibility in ultracold atomic experiment could be used in designing the spatially varying hopping amplitude, and thus, provides an ideal playground for realizing such a FFLO superfluid state. Furthermore, the existence of the pure FFLO phase could facilitate an unambiguous experimental observation of FFLO pairing in atomic systems free of the influence from other quantum phases at edges. The MSF which can be detected by the quantum polarization spectroscopy signal also provides a direct signature of finite center-of-mass momentum FFLO pairing.

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